	Operation	Inverse operation
1.	Interchange rows p, q	Interchange rows p, q
2.	multiply row p by $c \neq 0$	multiply row p by $\frac{1}{c}$
3.	Add k times row p to row q	Subtract k times row p from row q

LU-Factorisation

Lower triangular matrix $= \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = L$

$\rightarrow$ main diagonal

Upper triangular matrix $= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix} = U$

Special case diagonal matrix $= \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}$

Let A be an $m \times n$ matrix

$$A = LU$$

LU-factorisation: If the $n \times n$ matrix A can be written as the product of lower triangular matrix and upper triangular matrix then $A = LU$ is an LU-factorisation

Ex. $A = \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -10 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 14 \end{pmatrix}$

$\qquad\qquad\qquad\qquad\qquad L \qquad\qquad\qquad U$

How to find the LU-factorisation of an $n \times n$ matrix?

<u>Theorem</u>

If a square matrix can be reduced to row echelon form <u>without</u> using any row interchanges, then the matrix has an LU-factorisation

~~Steps~~ Ex $A = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix}$

<u>Steps</u>

$\boxed{1}$ $R_2 - 2R_1$, $R_3 - 3R_1$ $\Longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 12 & 16 \end{pmatrix}$

$\boxed{2}$ $R_3 - 4R_2 \Longrightarrow \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = U$

$\boxed{3}$ Divide each circled group by first number in the group to make it 1 (not negative 1)

$\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} = L$

Check: $LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 & 2 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 4 & 7 & 7 \\ 6 & 18 & 22 \end{pmatrix} = A$

Ex. $\begin{pmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$

$\begin{matrix} R_2 + R_1 \\ R_3 - 2R_1 \\ R_4 + 3R_1 \end{matrix} \Longrightarrow \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 10 & 4 & -9 \\ 0 & -16 & -11 & 18 \end{pmatrix} \xrightarrow[R_4 - 8R_1]{R_3 + 5R_2} \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & -3 & 2 \end{pmatrix}$

$\xrightarrow{R_4 - 3R_3} \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = U$

$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{pmatrix}$

Solve

$$3x_1 - 7x_2 - 2x_3 + 2x_4 = -9$$
$$-3x_1 + 5x_2 + x_3 = 5$$
$$6x_1 - 4x_2 - 5x_4 = 7$$
$$-9x_1 + 5x_2 - 5x_3 + 12x_4 = 11$$

$AX = B$ $\quad A = \begin{pmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{pmatrix}$ $\quad B = \begin{pmatrix} -9 \\ 5 \\ 7 \\ 11 \end{pmatrix}$

$\boxed{1}$ put $A = LU$

$\therefore LUX = B$

$\boxed{2}$ let $Y = UX$

$\therefore LY = B$ we get Y

$\boxed{3}$ $Y = UX$, we get X

$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{pmatrix}$ $\quad U = \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

$UX = Y$

$LY = B$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -9 \\ 5 \\ 7 \\ 11 \end{pmatrix}$$

$y_1 = -9 \quad y_2 = -4 \quad y_3 = 5 \quad y_4 = 1$

$Y = UX$

$$\begin{pmatrix} -9 \\ -4 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

$x_4 = -1 \qquad x_3 = -6 \qquad x_2 = 4 \qquad x_1 = 3$

3 Determinants

Every square matrix can be associated with a real number called its determinant

Ex. $A = \begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}$, $\det(A) = |A| = (3 \times 4) - (2 \times 5) = 2$

* The determinant can be +ve, −ve or zero

Minors and Cofactors

If A is a square matrix, then the minor M_{ij} of the element a_{ij} is the determinant of the matrix obtained by deleting the ith row and the jth column

Ex. $A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$ $\qquad$ $M_{11} = (-1 \times 1) - (2 \times 0) = -1$

$M_{31} = (2 \times 2) - (1 \times -1) = 5$

The Cofactor C_{ij} is given by

$$C_{ij} = (-1)^{i+j} M_{ij}$$

Find the minors of

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$$

$$M = \underbrace{\begin{pmatrix} -1 & -5 & 4 \\ 2 & -4 & -8 \\ 5 & -3 & -6 \end{pmatrix}}_{\text{minors}} \qquad C = \underbrace{\begin{pmatrix} -1 & 5 & 4 \\ -2 & -4 & 8 \\ 5 & 3 & -6 \end{pmatrix}}_{\text{Cofactors}}$$

$$B = \begin{pmatrix} 2 & -3 & 10 \\ 1 & 2 & -2 \\ 0 & 1 & -3 \end{pmatrix}$$

$$M = \begin{pmatrix} -4 & -3 & 1 \\ -1 & -6 & 2 \\ -14 & -14 & 7 \end{pmatrix} \qquad C = \begin{pmatrix} -4 & 3 & 1 \\ 1 & -6 & -2 \\ -14 & 14 & 7 \end{pmatrix}$$

general matrix of C $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

- If A is a square matrix, then the determinant of A is the sum of the entries in any row or column of A multiplied by their cofactors

Ex. Find the determinant of

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & 0 & 1 \end{pmatrix}$$

Take row 2, $a_{21} = 3$, $M_{21} = 2$, $C_{21} = -2$

$a_{22} = -1$, $M_{22} = -4$ $C_{22} = -4$

$a_{23} = 2$, $M_{23} = -8$ $C_{23} = 8$

$\det(A) = (3\times -2) + (-1\times -4) + (2\times 8) = 14$

OR Take column 3, $a_{13} = 1$, $M_{13} = 4$, $C_{13} = 4$

$a_{23} = 2$, $M_{23} = -8$, $C_{23} = 8$

$a_{33} = 1$, $M_{33} = -6$, $C_{33} = -6$

$\det(A) = (1\times 4) + (2\times 8) + (1\times -6) = 14$

How to evaluate the determinant of a 3x3 matrix?

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \Longrightarrow \begin{array}{ccccc} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{array}$$

$$\det(A) = (a_{11}\times a_{22}\times a_{33}) + (a_{12}\times a_{23}\times a_{31}) + (a_{13}\times a_{21}\times a_{32})$$
$$- (a_{31}\times a_{22}\times a_{13}) - (a_{32}\times a_{23}\times a_{11}) - (a_{33}\times a_{21}\times a_{12})$$

Ex find determinat of

$$A = \begin{pmatrix} 0 & 2 & 1 \\ 3 & -1 & 2 \\ 4 & -4 & 1 \end{pmatrix} \qquad \begin{array}{ccccc} 0 & 2 & 1 & 0 & 2 \\ 3 & -1 & 2 & 3 & -1 \\ 4 & -4 & 1 & 4 & -4 \end{array}$$

$\det(A) =$

$(0\times -1\times 1) + (2\times 2\times 4) + (1\times 3\times -4) - (4\times -1\times 1) - (-4\times 2\times 0) - (1\times 3\times 2)$

$= 0 + 16 - 12 + 4 \cancel{+8} - 6 = 2$

Special Case

If A is a triangular matrix, then its determinant is the pr[illegible]
of the entries on the main diagonal

Ex.

$$A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 \\ -5 & 6 & 1 & 0 \\ 1 & 5 & 3 & 3 \end{pmatrix}$$

$\det(A) = 2 \times -2 \times 1 \times 3 = -12$